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B. Sc. (Honrs) Part 1Paper 1

Subject Mathematics

Title/Heading of topic:Theory of equations

(fundamental theorem of algebra)

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1 Theory of equations

1.1. Polynomial Functions

Definition:

A function defined by

$f(x) = a_0x^n + a_1x^{n-1} + ... + a_n,$ *where* $a_o \neq 0$, n is a non negative integer and a_i (i = 0, 1….,n) are fixed complex numbers is called a **polynomial** of **degree** n in x. Then numbers $a_o, a_1, ..., a_n$ are called the **coefficients** of f.

If α is a complex number such that f(α)= 0, then α is called **zero** of the polynomial.

1.1.1 Theorem (Fundamental Theorem of Algebra)

Every polynomial function of degree $n \geq 1$ has at least one zero.

Remark:

Fundamental theorem of algebra says that, if f(x) = $a_0x^n + a_1x^{n-1} + ... + a_n$, where $a_0 \neq 0$ is the given polynomial of degree $n \geq 1$, then there exists a complex number α such that $a_0\alpha^n + a_1\alpha^{n-1} + ... + a_n = 0$.

We use the Fundamental Theorem of Algebra, to prove the following result.

1.1.2 Theorem

Every polynomial of degree n has n and only n zeroes.

Proof:

Let $f(x) = a_0x^n + a_1x^{n-1} + ... + a_n$, *where* $a_o \neq 0$, be a polynomial of degree $n \geq 1$.

By fundamental theorem of algebra, f(x) has at least one zero, let α_1 be that zero.

Then $(x - \alpha_1)$ is a factor of f(x).

Therefore, we can write:

f(x) = $(x - \alpha_1)Q_1(x)$, where $Q_1(x)$ is a polynomial function of degree n - 1.

If $n - 1 \geq 1$, again by Fundamental Theorem of Algebra, $Q_1(x)$ has at least one zero, say α_2.

Therefore, $f(x) = (x - \alpha_1)(x - \alpha_2)Q_2(x)$ *where* $Q_2(x)$ is a polynomial function of degree n - 2.

Repeating the above arguments, we get

$f(x) = (x - \alpha_1)(x - \alpha_2).....(x - \alpha_n)Q_n(x)$, where $Q_n(x)$ is a polynomial function of degree n - n = 0, i.e., $Q_n(x)$ is a constant.

Equating the coefficient of x^n on both sides of the above equation, we get

$Q_n(x) = a_o$.

Therefore, $f(x) = a_o(x - \alpha_1)(x - \alpha_2)....(x - \alpha_n)$.

If α is any number other than $\alpha_1, \alpha_2, ...\alpha_n$, then $f(x) \neq 0 \Rightarrow \alpha$ is not a zero of f(x).

Hence f(x) has n and only n zeros, namely $\alpha_1, \alpha_2, ..., \alpha_n$.

Note:

Let $f(x) = a_o x^n + a_1 x^{n-1} + ... + a_n ; a_o \neq 0$ be an n^{th} degree polynomial in x.

Then, $a_o x^n + a_1 x^{n-1} + ... + a_n = 0$ --------- (1)

is called a **polynomial equation** in x of degree n.

A number α is called a **root** of the equation (1) if α is a zero of the polynomial f(x).

So theorem (1.1.2)can also be stated as : "Every polynomial equation of degree n has n and only n roots".

1.1.3 Theorem

If the equation $a_o x^n + a_1 x^{n-1} + ... + a_n = 0$, where $a_o, a_1, ... a_n$ are real numbers $(a_o \neq 0)$, has a complex root $\alpha + i\beta$, then it also has a complex root $\alpha - i\beta$. (i.e., complex roots occur in conjugate pairs for a polynomial equation with real coefficients).

Proof:

Let $f(x) = a_o x^n + a_1 x^{n-1} + ... + a_n, a_o \neq 0$

Given that $\alpha + i\beta$ is a root of $f(x) = 0$.

Consider $(x - (\alpha + i\beta)(x - (\alpha - i\beta) = (x-\alpha)^2 + \beta^2$.

Divide f(x) by $(x-\alpha)^2 + \beta^2$.

Let Q(x) be the quotient and Ax + B be the remainder.

Then, $f(x) = [(x-\alpha)^2 + \beta^2] Q(x) + Ax + B$

$$= [(x - (\alpha + i\beta))(x - (\alpha - i\beta))] Q(x) + Ax + B$$

$$\Rightarrow f(\alpha + i\beta) = 0 + A(\alpha + i\beta) + B = A(\alpha + i\beta) + B = (A\alpha + B) + iA\beta$$

But $f(\alpha + i\beta) = 0$.

Equating real and imaginary parts, we see that $A\alpha + B = 0$ *and* $A\beta = 0$

But $\beta \neq 0 \Rightarrow A = 0$ *and so* $B = 0$

$\Rightarrow$ The remainder Ax + B is zero. i.e., $[(x - (\alpha + i\beta))(x - (\alpha - i\beta))]$ is a factor of f(x) i.e., $\alpha - i\beta$ is a root of $f(x) = 0$.

1.1.4. Theorem

In an equation with rational coefficients, the roots which are quadratic surds occur in conjugate pairs.

Proof:

Let $f(x) = a_o x^n + a_1 x^{n-1} + ... + a_n, a_o \neq 0$, be an n^{th} degree polynomial with rational coefficients.

Let $\alpha + \sqrt{\beta}$ is a root of $f(x) = 0$.

Divide f(x) by $[(x - (\alpha + \sqrt{\beta}))(x - (\alpha - \sqrt{\beta}))] = (x-\alpha)^2 - \beta$.

Let Q(x) be the quotient and Ax + B be the remainder.

Proceeding exactly as in the above theorem, we get Ax + B = 0.

Thus we conclude that $\alpha - \sqrt{\beta}$ is also a root of f(x) = 0.

1.1.5. Theorem

If the rational number p/q, a fraction in its lowest terms (so that p, q are integers prime to each other, $q \neq 0$) is a root of the equation $a_o x^n + a_1 x^{n-1} ... + a_n = 0$ where $a_0, a_1, ..., a_n$ are integers and $a_o \neq 0$, then p is a divisor of a_n and q is a divisor of a_o.

Proof:

Since p/q is a root the given polynomial equation, we have

$$a_o\left(p/q\right)^n + a_1\left(p/q\right)^{n-1} + ... + a_{n-1}\left(p/q\right) + a_n = 0$$

Multiplying by q^n, we get

$$a_o p^n + a_1 p^{n-1} q + ... + a_{n-1} p q^{n-1} + a_n q^n = 0 \quad \text{------------ (1)}$$

Dividing by p, we have

$$a_o p^{n-1} + a_1 p^{n-2} q + ... + a_{n-1} q^{n-1} = \frac{-a_n q^n}{p}$$

Now, the left side of the above equation is an integer and therefore $\frac{-a_n q^n}{p}$ is also must be an integer. Since p and q have no common factor, p must be a divisor of a_n.

Also, from (1),

$$a_1 p^{n-1} q + ... + a_{n-1} p q^{n-1} + a_n q^n = -a_o p^n$$

Dividing this expression by q, we get

$$a_1 p^{n-1} + ... + a_{n-1} p q^{n-2} + a_n q^{n-1} = \frac{-a_o p^n}{q}$$

Since the left side is an integer and since q does not divide p, q must be a divisor of a_0. This completes the proof.

Corollary

Every rational root of the equation $x^n + a_1 x^{n-1} + ... + a_n = 0$,where each a_i is an integer must be an integer.

Moreover, every such root must be a divisor of the constant a_n.

Proof:

This follows from the above theorem, by putting a_0 =1.

Multiple Roots

If a root α of f(x) = 0 repeats r times, then α is called an r-**multiple root**.

A 2- multiple root is usually called a **double root**.

For example, consider $f(x) = (x - 2)^3 (x - 5)^2 (x + 1)$.

Here 2 is a 3 - multiple root, 5 is a double root, and -1 is a single root of the equation f(x) = 0.

1.1.6. Theorem

If α is an r - multiple root of f(x) = 0 then α is an (r – 1) multiple root of $f^1(x) = 0$, where $f^1(x)$ is the derivative of f(x).

Proof:

Given that α is an r - multiple root of f(x) = 0.

Then $f(x) = (x-\alpha)^r \phi(x)$ *where* $\phi(\alpha) \neq 0$.

Now, by applying product rule of differentiation, we obtain:

$$f^1(x) = (x-\alpha)^r \phi^1(x) + \phi(x) \; r.(x-\alpha)^{r-1}$$

$$= (x-\alpha)^{r-1}[(x-\alpha)\phi^1(x) + r\phi(x)]$$

When $x = \alpha$, $(x-\alpha)\phi^1(x) + r\phi(x) = r\phi(\alpha) \neq 0$

$\Rightarrow \alpha$ is an (r – 1) multiple root of $f^1(x) = 0$.

Remark:

If α is an (r – 1) -multiple root of $f^1(x) = 0$, similarly as above, we can see that α will be an (r – 2) multiple root of $f^{11}(x) = 0$; (r – 3) - multiple root of $f^{111}(x) = 0$, and so on.

Solved Problems

1. Solve $x^4-4x^2+8x+35=0$, given $2+i\sqrt{3}$ is a root.

Solution :

Given that $2+i\sqrt{3}$ is a root of $x^4-4x^2+8x+35=0$; since complex roots occurs in conjugate pairs $2-i\sqrt{3}$ is also a root of it.

$\Rightarrow [x-(2+i\sqrt{3})][x-(2-i\sqrt{3})]=(x-2)^2+3=x^2-4x+7$ is a factor of the given polynomial.

Dividing the given polynomial by this factor, we obtain the other factor as x^2+4x+5.

The roots of $x^2+4x+5=0$ are given by $\frac{-4\pm\sqrt{16-20}}{2}=-2\pm i$.

Hence the roots of the given polynomial are $2+i\sqrt{3},\ 2-i\sqrt{3},\ -2+i$ *and* $-2-i$.

2. Solve $x^4-5x^3+4x^2+8x-8=0$, given that one of the roots is $1-\sqrt{5}$.

Solution:

Since quadratic surds occur in conjugate pairs as roots of a polynomial equation, $1+\sqrt{5}$ is also a root of the given polynomial.

$\Rightarrow [x-(1-\sqrt{5})][x-(1+\sqrt{5})]=(x-1)^2-5=x^2-2x-4$ is a factor.

Dividing the given polynomial by this factor, we obtain the other factor as x^2-3x+2.

Also, $x^2-3x+2=(x-2)(x-1)$

Thus the roots of the given polynomial equation are $1+\sqrt{5}, 1-\sqrt{5}, 1, 2$.

3. Find a polynomial equation of the lowest degree with rational coefficients having $\sqrt{3}$ and $1-2i$ as two of its roots.

Solution:

Since quadratic surds occur in pairs as roots, $-\sqrt{3}$ is also a root.

Since complex roots occur in conjugate pairs, $1+2i$ is also a root of the required polynomial equation. Therefore the desired equation is given by

$$(x-\sqrt{3})(x+\sqrt{3})(x-(1-2i)(x-(1+2i))=0$$

$$\text{i.e., } x^4-2x^3+2x^2+6x-15=0$$

4. Solve $4x^5+x^3+x^2-3x+1=0$, given that it has rational roots.

Solution:

Let $f(x)=4x^5+x^3+x^2-3x+1$.

By theorem (1.1.5.), any rational root p/q (in its lowest terms) must satisfy the condition that, p is divisor of 1 and q is positive divisor of 4.

So the possible rational roots are $\pm 1, \pm \frac{1}{2}, \pm \frac{1}{4}$.

Note that $f(-1) = 0$, $f(\frac{1}{2}) = 0$. But $f(1) \neq 0$, $f(-\frac{1}{2}) \neq 0$, $f(\frac{1}{4}) \neq 0$ and $f(-1/4) \neq 0$.

Since $f(-1) = 0$ and $f(\frac{1}{2}) = 0$, we see that $(x + 1)$ and $(x - \frac{1}{2})$ are factors of the given polynomial. Also by factorizing , we find that

$$f(x) = (x - \tfrac{1}{2})(x + 1)(4x^3 - 2x^2 + 4x - 2)$$

Note that $x = \frac{1}{2}$ is a root of the third factor, if we divide $4x^3 - 2x^2 + 4x - 2$ by $x - \frac{1}{2}$, we obtain

$$f(x) = (x - \tfrac{1}{2})^2 (x + 1)(4x^2 + 4)$$
$$= 4(x - \tfrac{1}{2})^2 (x + 1)(x^2 + 1)$$

Hence the roots of $f(x) = 0$, are $\frac{1}{2}, \frac{1}{2}, -1, \pm i$.

5. Solve $x^3 - x^2 - 8x + 12 = 0$, given that has a double root.

Solution:

Let $f(x) = x^3 - x^2 - 8x + 12$

Differentiating, we obtain:

$$f^1(x) = 3x^2 - 2x - 8.$$

Since the multiple roots of $f(x) = 0$ are also the roots of $f^1(x) = 0$, the product of the factors corresponding to these roots will be the g.c.d of $f(x)$ and $f^1(x)$. Let us find the g.c.d of $f(x)$ and $f^1(x)$.

$3x$	$3x^2 - 2x - 8$	$x^3 - x^2 - 8x + 12$	
	$3x^2 - 6x$	3	
4	$4x - 8$	$3x^3 - 3x^2 - 24x + 36$	x
	$4x - 8$	$3x^3 - 2x^2 - 8x$	
0	0	$-x^2 - 16x + 36$	
		3	
		$-3x^2 - 48x + 108$	-1
		$-3x^2 + 2x + 8$	
		-50 | $-50x + 100$	
		$x - 2$	

Therefore, g.c.d $= (x - 2)$

$\Rightarrow f(x)$ has a factor $(x - 2)^2$.

Also, $f(x) = (x - 2)^2 (x + 3)$

Thus the roots are $2, 2, -3$.

6. Show that the equation $x^3 + qx + r = 0$ has two equal roots if $27r^2 + 4q^3 = 0$.

Solution:

Let $f(x) = x^3 + qx + r$ ------------- (1)

Differentiating, we obtain: $f^1(x) = 3x^2 + q$ ----------------- (2)

Given that $f(x) = 0$ has two equal roots, i.e., it has a double root, say α.

Then α is a root of both $f(x) = 0$ and $f^1(x) = 0$.

From the 2nd equation, we obtain $\alpha^2 = -q/3$

Now the first equation can be written as: $\alpha(\alpha^2 + q) + r = 0$

i.e., $\alpha(-q/3 + q) + r = 0 \Rightarrow \alpha = \frac{-3r}{2q}$

Squaring and simplifying, we obtain: $27r^2 + 4q^3 = 0$
